Introduction to NTRU Public Key Cryptosystem[†]

NTRUEncrypt

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Outline

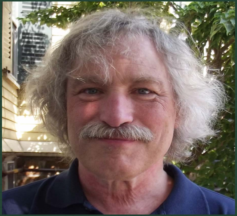
1. Introduction
2. Convolution Polynomial Rings
3. Multiplicative Inverse
4. `NTRUEncrypt`
5. Security
6. Performance
7. Conclusion

NTRU

- NTRU: *Nth-degree TRUncated polynomial ring* (pronounced *en-trū*)

NTRU

- `NTRU`: *Nth-degree TRUncated polynomial ring* (pronounced *en-trū*)
- A public key cryptosystem [HPS98] invented in early 1996 by



Hoffstein

Pipher

Silverman

Ring of Convolution Polynomials

Definition

The ring of convolution polynomials of rank N[1] is the quotient ring

$$R = \frac{\mathbb{Z}[x]}{\langle x^N - 1 \rangle}$$

[1] a.k.a. N-th truncated polynomial ring

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The ring of convolution polynomials modulo q of rank N is the quotient ring

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- Every element of R has a unique representation of the form

$$a_0 + a_1x + a_2x^2 + \cdots + a_{N-1}x^{N-1} = \sum_{i=0}^{N-1} a_i x^i \text{ or } \mathbf{a} = (a_0, \cdots, a_{N-1})$$

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$$x^k = x^r.$$
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- Polynomials in R_q can also be uniquely identified in the same way.

Operations of Convolution Polynomial Rings

Every ring has two operations, i.e, addition and multiplication.

- **Addition** of polynomials correspond to the usual addition of vectors,

$$a(x) + b(x) \leftrightarrow (a_0 + b_0, a_1 + b_1, a_2 + b_2, \dots, a_{N-1} + b_{N-1}).$$

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- **Multiply** two polynomials mod $x^N - 1$, i.e., replace x^k with $x^{k \mod N}$.

$$\mathbf{c} = \mathbf{a} \star \mathbf{b}, \quad c_i = \sum_{j=0}^{N-1} a_j b_{i-j}$$

Example

Let $N = 3$ and $a(x) = 1 + 3x + x^2$, and $b(x) = -4 + x + 2x^2$. Then

$$\begin{aligned} a(x) + b(x) &= (1-4) + (3+1)x + (1+2)x^2 = -3 + 4x + 3x^2 \\ a(x) \star b(x) &= -4 - 11x + x^2 + 7x^3 + 2x^4 \\ &= -4 - 11x + x^2 + 7 + 2x \\ &= 3 - 9x + x^2 \in R = \frac{\mathbb{Z}[x]}{\langle x^3 - 1 \rangle} \\ &= 3 + 5x + x^2 \in R_7 = \frac{\mathbb{Z}_7[x]}{\langle x^3 - 1 \rangle}. \end{aligned}$$

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$$\begin{aligned} \mathbf{a} \star \mathbf{b} = [a_0 \quad a_1 \quad a_2] \begin{bmatrix} b_0 & b_1 & b_2 \\ b_2 & b_0 & b_1 \\ b_1 & b_2 & b_0 \end{bmatrix} = [1 \quad 3 \quad 1] \begin{bmatrix} -4 & 1 & 2 \\ 2 & -4 & 1 \\ 1 & 2 & -4 \end{bmatrix} \\ = [3 \quad -9 \quad 1] \end{aligned}$$

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Let $N = 3$ and $a(x) = 1 + 3x + x^2$, and $b(x) = -4 + x + 2x^2$. Then

$$\begin{aligned}
a(x) + b(x) =& (1-4) + (3+1)x + (1+2)x^2 = -3 + 4x + 3x^2 \\
a(x) \star b(x) =& -4 - 11x + x^2 + 7x^3 + 2x^4 \\
=& -4 - 11x + x^2 + 7 + 2x \\
=& 3 - 9x + x^2 \in R = \frac{\mathbb{Z}[x]}{\langle x^3 - 1 \rangle} \\
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$$\mathbf{a} \star \mathbf{b} = [a_0 \quad a_1 \quad a_2] \begin{bmatrix} b_0 & b_1 & b_2 \\ b_2 & b_0 & b_1 \\ b_1 & b_2 & b_0 \end{bmatrix} = [1 \quad 3 \quad 1] \begin{bmatrix} -4 & 1 & 2 \\ 2 & -4 & 1 \\ 1 & 2 & -4 \end{bmatrix}$$
$$= [3 \quad -9 \quad 1]$$

A polynomial multiplication takes N^2 multiplications.

Convolution Polynomial Rings in Sage I

- Generate $R = \frac{\mathbb{Z}[x]}{\langle x^7 - 1 \rangle}$:

```
N = 7
ZX.<X> = PolynomialRing(ZZ)
R.<x> = ZX.quotient(X^N - 1); R
Univariate Quotient Polynomial Ring in x over
Integer Ring with modulus X^7 - 1
```

- Generate $R_3 = \frac{\mathbb{Z}_3[x]}{\langle x^7 - 1 \rangle}$

```
N, q = 7, 3
ZqX.<X> = PolynomialRing(Zmod(q))
Rq.<x> = ZqX.quotient(X^N - 1); Rq
Univariate Quotient Polynomial Ring in x over
Ring of integers modulo 3 with modulus X^7 + 2
```

Convolution Polynomial Rings in Sage II

- Choose two elements at random from R, and multiply them:

```
[f, g] = [Rq.random_element() for _ in range(2)]
print("(f, g) = ", (f, g))
print("f*g = ", f*g)
(f, g) =  (2*x^6 + 2*x^4 + x^3, 2*x^6 + x^2 + 2*x)
f*g =  2*x^6 + 2*x^4 + x^3 + 2*x^2 + 2*x + 1
```

- Lift $f \in R_3 = \frac{\mathbb{Z}_3[X]}{\langle X^7 - 1 \rangle}$ into $\mathbb{Z}_3[X]$

```
print(f.parent())
Univariate Quotient Polynomial Ring in x over
Ring of integers modulo 3 with modulus X^7 + 2

f = f.lift()
print(f.parent())
Univariate Polynomial Ring in X over
Ring of integers modulo 3
```

Multiplicative Inverse I

$f(x) \in R_q$ has a multiplicative inverse if and only if

$$\gcd(f(x), x^N - 1) = 1 \in \mathbb{Z}_q[x].$$

If so, then the inverse $f(x)^{-1} \in R_q$ can be computed using the extended Euclidean algorithm to find polynomials $u(x), v(x) \in \mathbb{Z}_q[x]$ satisfying

$$f(x) \star u(x) + (x^N - 1) \star v(x) = 1.$$

Then $f^{-1}(x) = u(x) \in R_q$.

Multiplicative Inverse II

- You can simply compute the inverse via SageMath[The21] (if it exists!)

```
reset()
N, q = 7, 4
Zx.<X> = ZZ[]
f = X^6 - X^4 + X^3 + X^2 -1
Zq.<a> = PolynomialRing(Zmod(q))
f = Zq(f) # Moving f from Zx[x] into Zq[a]
print("gcd(f, a^N - 1) = ", f.gcd(a^N - 1))
f_inv = f.inverse_mod(a^N - 1); f_inv(a=X)

gcd(f, a^N - 1) =  1
X^5 + 3*X^4 + 3*X^3 + 2*X^2
```

- Check to see if the multiplication of $f \star f^{-1} = 1 \mod q$?

```
Zq(f*f_inv).mod(a^N - 1)

1
```

NTRUEncrypt

Parameters: $N, p, q, (p, q) = 1$. E.g., $N = 401, p = 3, q = 2048$

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Definition (Centered modular reduction)

For an odd integer n and integers a and b, define

$$a \bmod n = b \text{ if } a \equiv b \bmod n \text{ and } -\frac{n-1}{2} \le b \le \frac{n}{2}.$$

For example $a \bmod 5 \in \{-2,-1,0,1,2\}$, whereas a $\bmod 5 \in \{0,1,2,3,4\}$.

NTRUEncrypt

- Key-Generation:
  - Choose $F(x), G(x) \in R$ s.t. $\mathbf{F}, \mathbf{G} \in \{-1, 0, 1\}^N$.

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  - $f(x) = 1 + pF(x)$, compute $f^{-1}(x)$
  - $g(x) = pG(x)$
  - Compute $h(x) = f^{-1}(x) \star g(x) \bmod q$

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- Enc:
  - Plaintext $\mathbf{m} \in \{-1, 0, 1\}^N$
  - Choose $\mathbf{r} \in \{-1, 0, 1\}^N$ uniformly at random
  - Ciphertext $\mathbf{y} = \mathbf{r} \star \mathbf{h} + \mathbf{m} \bmod q$

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- Dec:
  - Compute $\mathbf{a} = \mathbf{f} \star \mathbf{y} \bmod q$
  - Compute $\mathbf{m}' = \mathbf{a} \bmod p$

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$$\begin{aligned}\mathbf{a} &= \mathbf{f} \star \mathbf{r} \star \mathbf{h} + \mathbf{f} \star \mathbf{m} \bmod q \quad (\mathbf{y} = \mathbf{r} \star \mathbf{h} + \mathbf{m} \bmod q) \\ &= \mathbf{f} \star \mathbf{r} \star \mathbf{f}^{-1} \star \mathbf{g} + \mathbf{f} \star \mathbf{m} \bmod q \quad (\mathbf{h} = \mathbf{f}^{-1} \star \mathbf{g} \bmod q) \\ &\equiv \mathbf{r} \star \mathbf{g} + \mathbf{f} \star \mathbf{m} \bmod q\end{aligned}$$

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  - Compute $\mathbf{m}' = \mathbf{a} \bmod p$
    If the coefficients of $\mathbf{r} \star \mathbf{g} + \mathbf{f} \star \mathbf{m}$ lie in the interval

$$\left[-\frac{q-1}{2}, \frac{q}{2} \right],$$

    which holds with high probability. In such cases,

$$\mathbf{a} = \mathbf{f} \star \mathbf{y} \bmod q = \mathbf{r} \star \mathbf{g} + \mathbf{f} \star \mathbf{m}$$

$$\begin{aligned} \mathbf{m}' = (\mathbf{f} \star \mathbf{y} \bmod q) \quad \bmod p &= \mathbf{r} \star \mathbf{g} + \mathbf{f} \star \mathbf{m} \bmod p \\ &\equiv \mathbf{m} \bmod p. \quad (\mathbf{g} = \mathbf{pG}, \mathbf{f} = \mathbf{1} + \mathbf{pF}) \end{aligned}$$

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$$\begin{aligned} \mathbf{a} &= \mathbf{f} \star \mathbf{r} \star \mathbf{h} + \mathbf{f} \star \mathbf{m} \bmod q \quad (\mathbf{y} = \mathbf{r} \star \mathbf{h} + \mathbf{m} \bmod q) \\ &= \mathbf{f} \star \mathbf{r} \star \mathbf{f}^{-1} \star \mathbf{g} + \mathbf{f} \star \mathbf{m} \bmod q \quad (\mathbf{h} = \mathbf{f}^{-1} \star \mathbf{g} \bmod q) \\ &\equiv \mathbf{r} \star \mathbf{g} + \mathbf{f} \star \mathbf{m} \bmod q \end{aligned}$$

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$$\begin{aligned} \mathbf{m}' = (\mathbf{f} \star \mathbf{y} \bmod q) \quad \bmod p &= \mathbf{r} \star \mathbf{g} + \mathbf{f} \star \mathbf{m} \bmod p \\ &\equiv \mathbf{m} \bmod p. \quad (\mathbf{g} = \mathbf{pG}, \mathbf{f} = \mathbf{1} + \mathbf{pF}) \end{aligned}$$

    Therefore, $\mathbf{m}' = \mathbf{m} =$. The ciphertext is decrypted correctly.

How does the decryption work?

If an attacker decrypts $\mathbf{y}$ with a $\mathbf{f}'$ where $\mathbf{f}' \neq \mathbf{f}$, can she/he recover the plaintext polynomial $\mathbf{m}$?

Decrypt with f'

Decrypt with $\mathbf{f}'$

- Dec:
  - Compute $\mathbf{a} = \mathbf{f}' \star \mathbf{y} \bmod q$

$$\mathbf{a} = \mathbf{f}' \star \mathbf{r} \star \mathbf{h} + \mathbf{f}' \star \mathbf{m} \bmod q \quad (\mathbf{y} = \mathbf{r} \star \mathbf{h} + \mathbf{m} \bmod q)$$

$$= \mathbf{f}' \star \mathbf{r} \star \mathbf{f}^{-1} \star \mathbf{g} + \mathbf{f}' \star \mathbf{m} \bmod q \quad (\mathbf{h} = \mathbf{f}^{-1} \star \mathbf{g} \bmod q)$$

$$\equiv \mathbf{r} \star \mathbf{g} + \mathbf{f} \star \mathbf{m} \bmod q$$

Decrypt with $\mathbf{f}'$

- Dec:
  - Compute $\mathbf{a} = \mathbf{f}' \star \mathbf{y} \bmod q$

$$\mathbf{a} = \mathbf{f}' \star \mathbf{r} \star \mathbf{h} + \mathbf{f}' \star \mathbf{m} \bmod q \quad (\mathbf{y} = \mathbf{r} \star \mathbf{h} + \mathbf{m} \bmod q)$$
$$= \mathbf{f}' \star \mathbf{r} \star \mathbf{f}^{-1} \star \mathbf{g} + \mathbf{f}' \star \mathbf{m} \bmod q \quad (\mathbf{h} = \mathbf{f}^{-1} \star \mathbf{g} \bmod q)$$
$$\equiv \mathbf{r} \star \mathbf{g} + \mathbf{f} \star \mathbf{m} \bmod q$$

  - Compute $\mathbf{m}' = \mathbf{a} \bmod p$

    If the following equation holds, the attacker can recover $\mathbf{m}$.

$$\mathbf{a} = \mathbf{f}' \star \mathbf{y} \bmod q = \mathbf{f}'' \star \mathbf{r} \star \mathbf{g} + \mathbf{f}' \star \mathbf{m}$$

where $\mathbf{f}'' = \mathbf{f}' \star \mathbf{f}^{-1} \bmod q$ Recall $\mathbf{g} = \mathbf{pG}, \mathbf{f} = \mathbf{1} + \mathbf{pF}$

$\mathbf{f}' \star \mathbf{m}$ and $\mathbf{r} \star \mathbf{g}$ still have small coefficients, whereas $\mathbf{f}'' \star \mathbf{r} \star \mathbf{g}$ is likely to have large coefficients.

Example I

Suppose $N = 11, p = 3$ and $q = 23$.

Key-Generation:

- Choose $F(x), G(x) \in R$ s.t. $\mathbf{F}, \mathbf{G} \in \{-1, 0, 1\}^N$.
  $F(x) = x^{10} - x^9 + x^8 - x^4 - x^2 + x$
  $f(x) = 3x^{10} - 3x^9 + 3x^8 - 3x^4 - 3x^2 + 3x + 1 \leftarrow f(x) = 1 + pF(x)$
  $f^{-1}(x) = -11x^{10} + 7x^9 - 8x^8 + 2x^7 + 6x^6 - x^5 - 2x^4 - 3x^3 - 3x^2 - 11x + 2$
  $G(x) = x^9 - x^8 - x^7 + x^6 + x^4 - 1,$
  $g(x) = 3x^9 - 3x^8 - 3x^7 + 3x^6 + 3x^4 - 3 \leftarrow g(x) = pG(x)$
- Compute $h(x) = f^{-1}(x) \star g(x) \bmod q$
  $h(x) = 7x^{10} - 8x^9 + 3x^8 - 10x^6 - 8x^5 - 6x^3 - 8x^2 + 4x + 3$
- PK: $h(x)$, SK: $f(x)$

Example II

Enc:

- Plaintext $\mathbf{m} \in \{-1, 0, 1\}^N$
  $m(x) = x^{10} - x^5 + x^3 - 1$
- Choose $\mathbf{r} \in \{-1, 0, 1\}^N$ uniformly at random
  $r(x) = x^9 + x^7 - x^6 - x^5 - x^4 + x^2.$
- Ciphertext $\mathbf{y} = \mathbf{r} \star \mathbf{h} + \mathbf{m} \bmod q$
  $y(x) = -3x^{10} + 9x^9 + -8x^8 - 3x^7 + 11x^6 - 6x^5 + 6x^4 - 5x^3 - 2x^2 + 1$

Example III

Dec:

- Compute $\mathbf{a} = \mathbf{f} \star \mathbf{y} \bmod q = \mathbf{f} \star \mathbf{r} \star \mathbf{f}^{-1} \star \mathbf{g} + \mathbf{f} \star \mathbf{m} \bmod q$
  $\mathbf{f} \star \mathbf{y} = 12x^{10} - 29x^8 + 3x^7 + 23x^6 + 45x^5 - 66x^4 + 67x^3 - 83x^2 + 63x - 35 \in R$
  $a(x) = -11x^{10} - 6x^8 + 3x^7 - x^5 + 3x^4 - 2x^3 + 9x^2 - 6x + 11 \in R_q$
- Compute $\mathbf{m}' = \mathbf{a} \bmod p$
  Coefficients of $a(x)$ all lie in the interval $[-11, 11]$. Applying mod 3 we have

$$m'(x) = x^{10} - x^5 + x^3 - 1 = m(x).$$

- Check
  $\mathbf{r} \star \mathbf{g} + \mathbf{f} \star \mathbf{m} = -11x^{10} - 6x^8 + 3x^7 - x^5 + 3x^4 - 2x^3 + 9x^2 - 6x + 11 \in R$

Example IV

Dec with incorect secret key $\mathbf{f}' = 1 + 3(x^9 - x^8 - x^6 - x^5 + x^3 + 1)$

- Compute $\mathbf{a} = \mathbf{f}' \star \mathbf{y} \bmod q = \mathbf{f}' \star \mathbf{r} \star \mathbf{f}^{-1} \star \mathbf{g} + \mathbf{f}' \star \mathbf{m} \bmod q$
  $a(x) = 8x^{10} + 5x^8 + 2x^7 - 2x^6 - 9x^5 + 2x^4 - 2x^3 + 6x^2 - 7x - 3 \in R_q$
- Compute $\mathbf{m}' = \mathbf{a} \bmod p$
  $m'(x) = -x^{10} - x^8 - x^7 + x^6 - x^4 + x^3 - x \neq m(x).$
- Check
  $\mathbf{f}'' = \mathbf{f}' \star \mathbf{f}^{-1} \bmod q = -7x^{10} - 9x^9 + 4x^8 - 4x^7 + 6x^6 - 7x^5 - 3x^4 + 3x^3 + 2x^2 - 11x + 4 \in R_q$
  $\mathbf{f}' \star \mathbf{m} = 7x^{10} - 6x^9 - 3x^7 + 6x^6 - 4x^5 - 3x^4 - 2x^3 + 6x^2 + 3x - 4 \in R$
  $\mathbf{r} \star \mathbf{g} = -9x^{10} - 9x^9 + 3x^6 + 3x^5 - 3x^3 + 6x^2 + 3x + 6$
  $\mathbf{f}' \star \mathbf{f}^{-1} \star \mathbf{r} \star \mathbf{g} = 24x^{10} + 213x^9 - 87x^8 - 18x^7 + 15x^6 - 51x^5 + 51x^4 - 69x^3 - 138x^2 - 33x + 93$
  $\mathbf{f}'' \star \mathbf{r} \star \mathbf{g}$ has large coefficients compared to $q/2$.

Conditions for parameters

- Each of $\mathbf{F}, \mathbf{G}, \mathbf{r}, \mathbf{m}$ have (roughly) $\frac{1}{3}$ of their coefficients equal to each of $-1, 0$ and 1.
  - Related to the security of the scheme.
- q should be large compared to N.
  - To ensure the decryption is correct with high probability.

What is the hard math problem behind NTRU?

- **Lattice reduction**
  - Same problem that breaks the knapsack!
- If attacker can determine $\mathbf{f}^{-1}$ or $\mathbf{g}$, from $\mathbf{h}$, she gets the private key.

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The NTRU Key Recovery Problem[HPSS08]

Given $h(x)$, find **ternary** polynomials $f(x)$ and $g(x)$ satisfying $f(x) \star h(x) = g(x) \mod q$ where coefficients of $f(x)$ and $g(x)$ lie in $\{-p, 0, p\}$.

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- Recall $\mathbf{h} = \mathbf{f}^{-1} \star \mathbf{g} \mod q$

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- Recall $\mathbf{h} = \mathbf{f}^{-1} \star \mathbf{g} \mod q$
- Equivalently, $\mathbf{f} \star \mathbf{h} \equiv \mathbf{g} \mod q$. I.e., there exists some integer vector $\mathbf{t}$ such that

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- It seems that $(\mathbf{f}, \mathbf{g})$ is the shortest vector in the lattice $\mathcal{L}$.

NTRU and SVP

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- There is no proof that breaking `NTRUEncrypt` is as hard as solving the **Shortest Vector Problem** or the **Closest Vector Problem**.
- In 2013, Damien Stehle and Ron Steinfeld created a provably secure version of NTRU [SS13].
- The European Union's PQCRYPTO project (Horizon 2020 ICT-645622) is evaluating the provably secure Stehle–Steinfeld version of NTRU as a potential European standard. However, the Stehle-Steinfeld version of NTRU is "significantly less efficient than the original scheme."

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- Promising PQC candidate

The National Institute of Standards and Technology wrote in a 2009 survey that "[there] are viable alternatives for both public key encryption and signatures that are not vulnerable to Shor's Algorithm" and "[of] the various lattice based cryptographic schemes that have been developed, the NTRU family of cryptographic algorithms appears to be the most practical".

Conclusion

- A lattice-based public key cryptosystem
- Its security relies on difficulty of `SVP` problem
- Has evolved since its introduction
- Considered theoretically sound
- Unlike `RSA` and `ECC`, `NTRU` is **not** known to be vulnerable against quantum computer based attack
- It has been standardized (IEEE Std 1363.1, X9.98)

Thanks for your attention!

Question?

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